Principles of Communications

EES 351

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2. Frequency-Domain Analysis

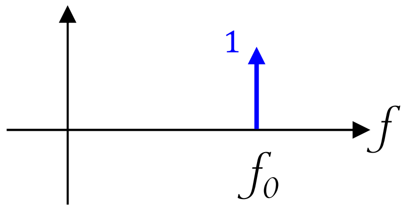
Part C

Fourier Transform Pairs (1)

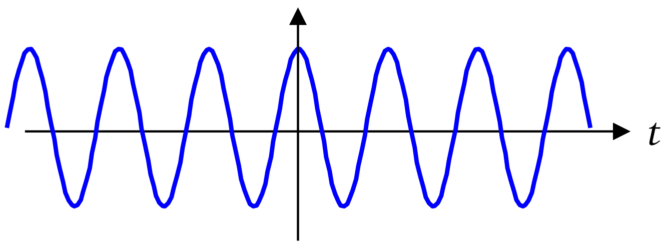
Time Domain | Frequency Domain

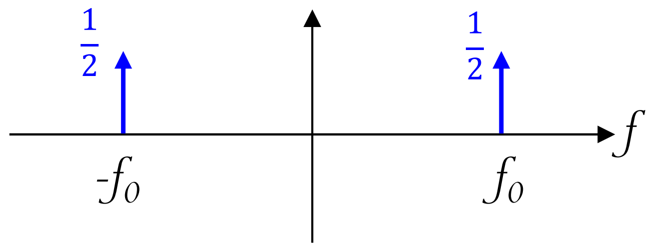
$$g(t) = \int_{-\infty}^{\infty} G(f)e^{j2\pi ft}df \overset{\mathcal{F}}{\rightleftharpoons} G(f) = \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}dt$$

$$e^{j2\pi f_0 t} \overset{\mathcal{F}}{\rightleftharpoons} \delta(f - f_0)$$



$$\frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2} = \cos(2\pi f_0 t) \overset{\mathcal{F}}{\rightleftharpoons} \frac{1}{2}\delta(f - (-f_0)) + \frac{1}{2}\delta(f - f_0)$$

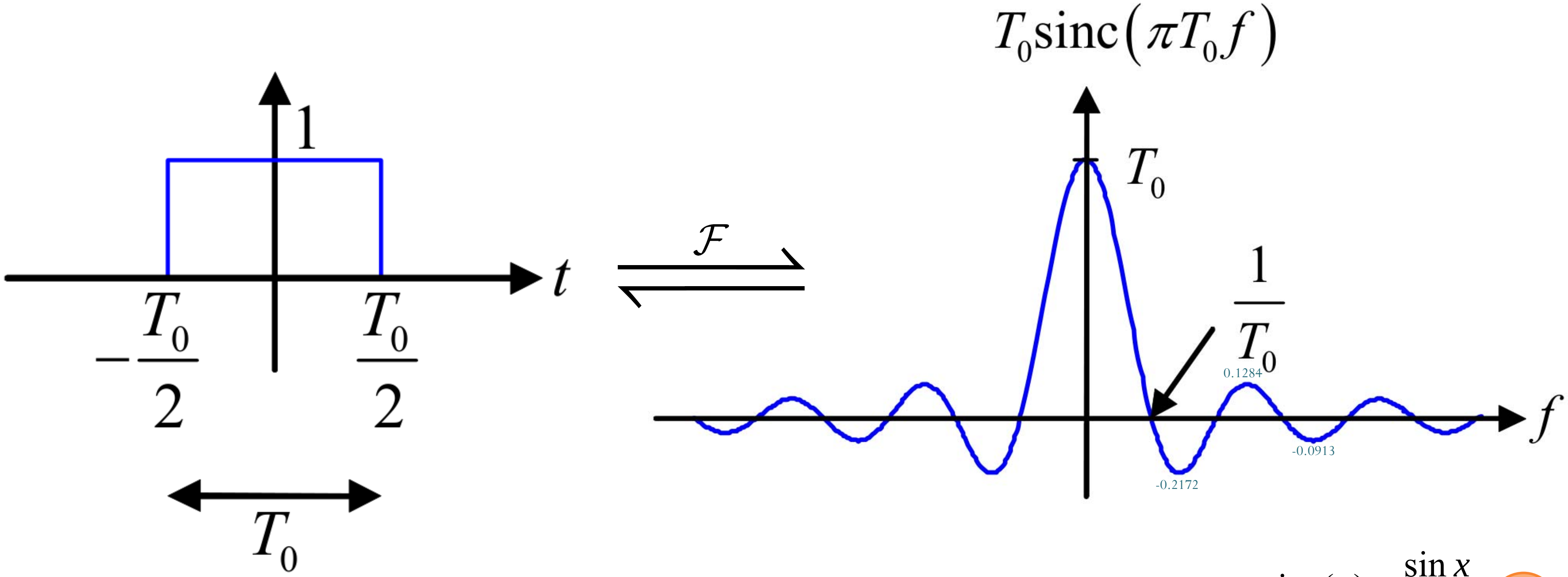




Fourier Transform Pairs (2)

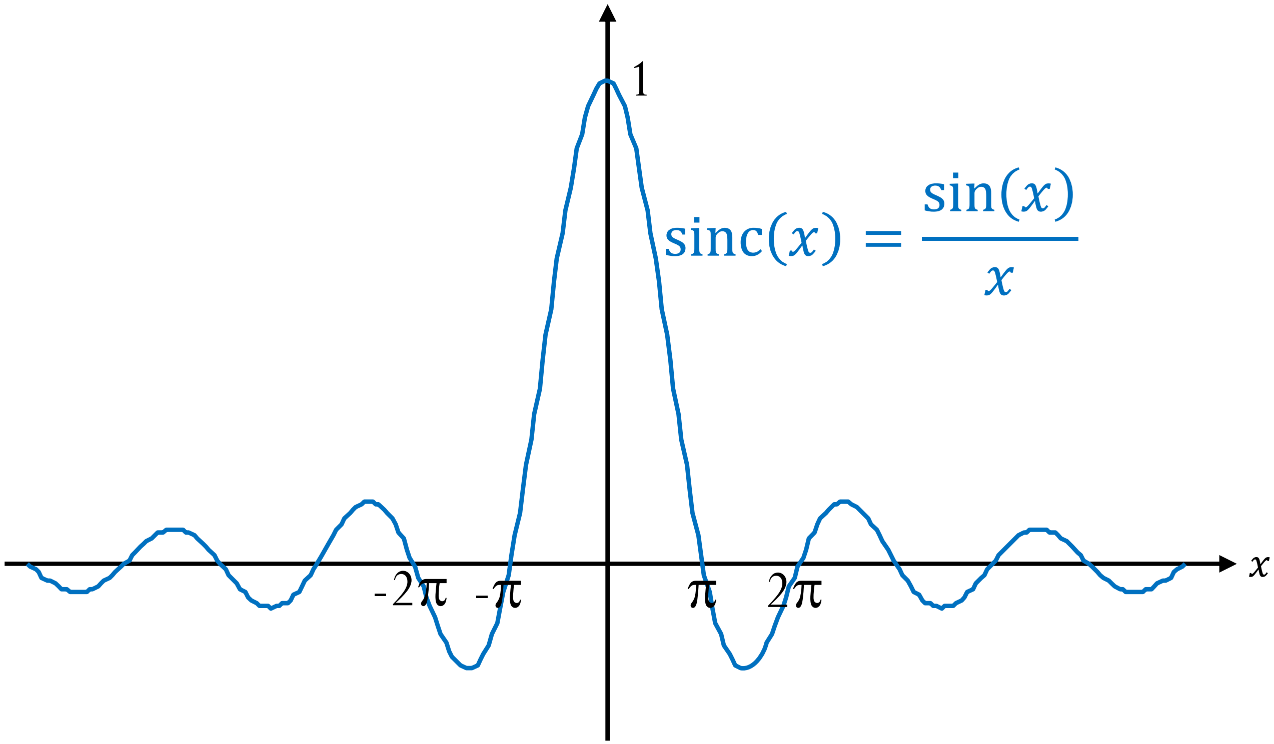
Time Domain | Frequency Domain

$$g(t)=\int_{-\infty}^{\infty} G(f)e^{j2\pi ft}df \overset{\mathcal{F}}{\rightleftharpoons} G(f)=\int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}dt$$

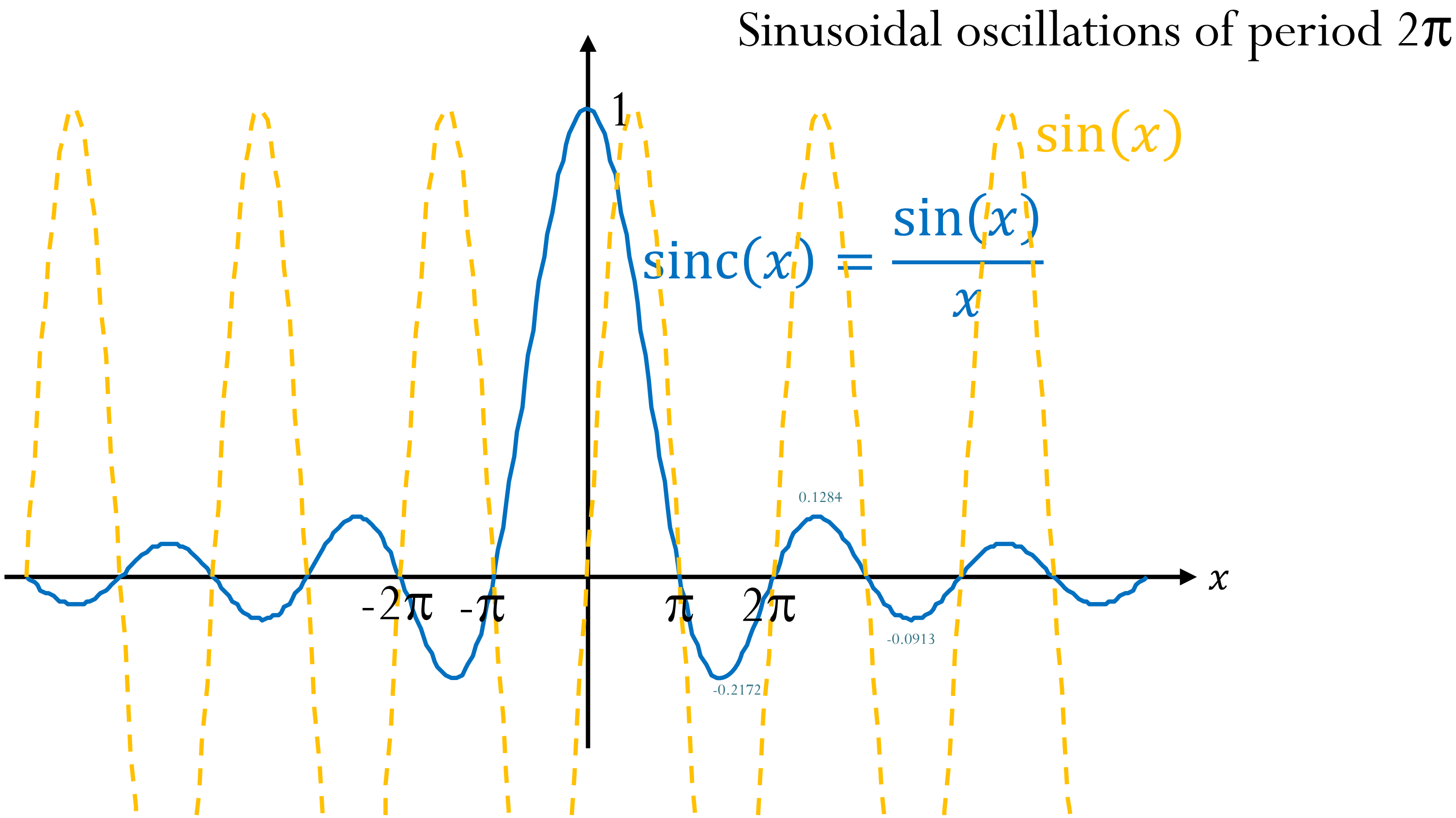


$$\text{sinc}(x)=\frac{\sin x}{x}$$

sinc function



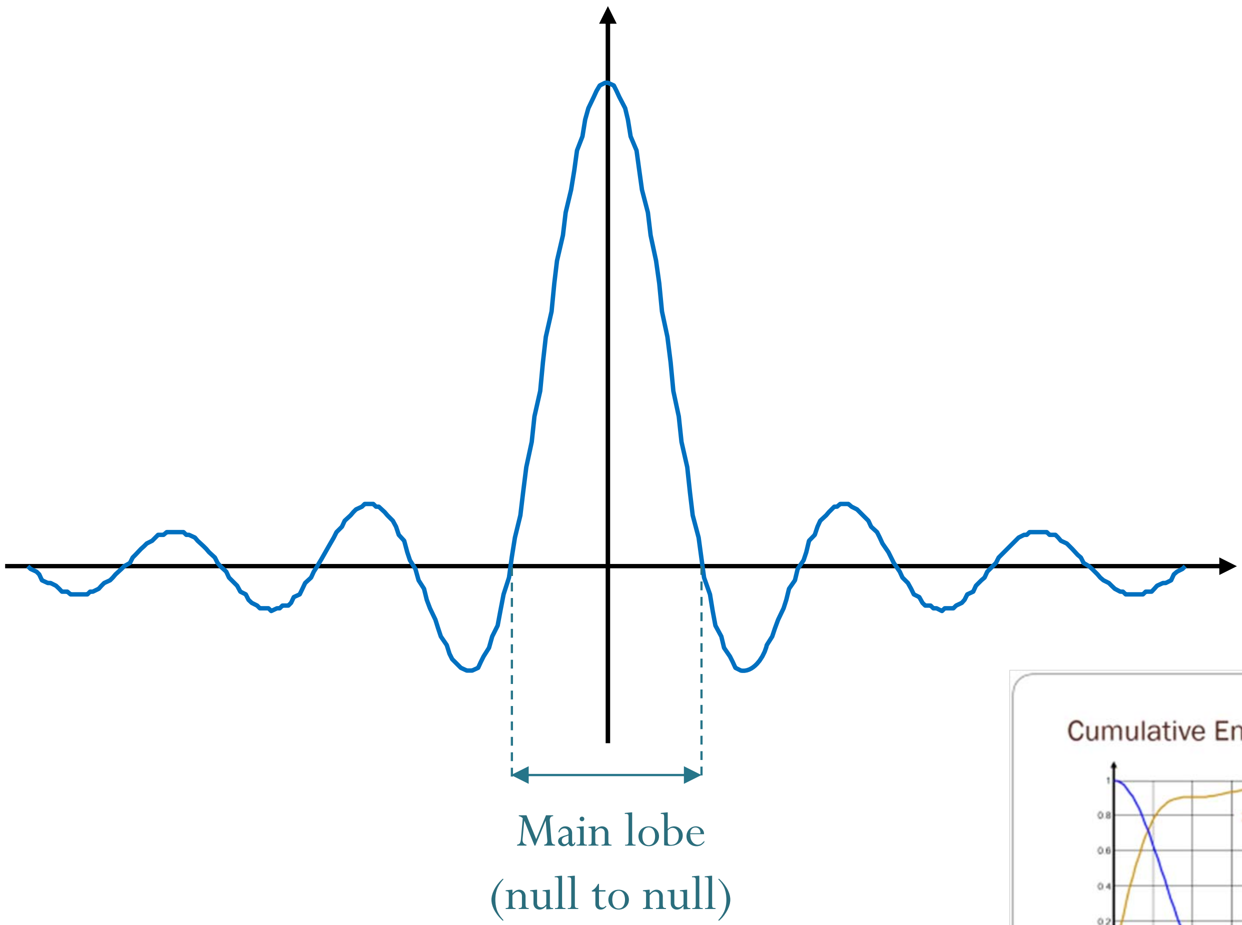
sinc function

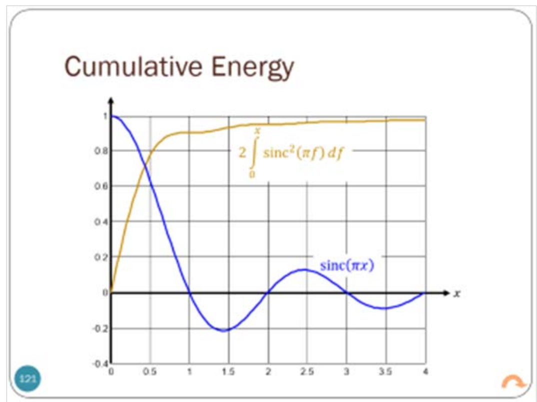


Zero crossings are at all non-zero integer multiples of π because $\sin(x) = 0$.

As $x \to 0$, we have $\frac{0}{0}$. Using L'Hospital's Rule, we set $\text{sinc}(0) \equiv 1$.

sinc function

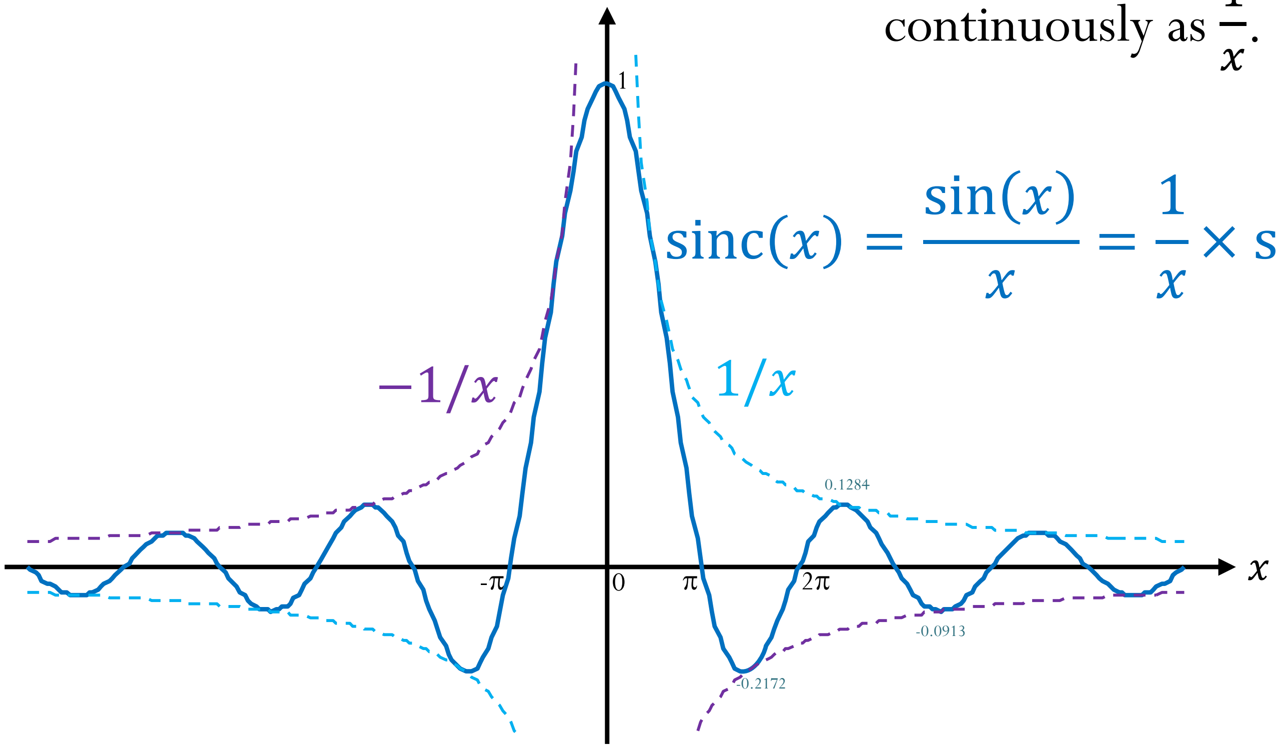




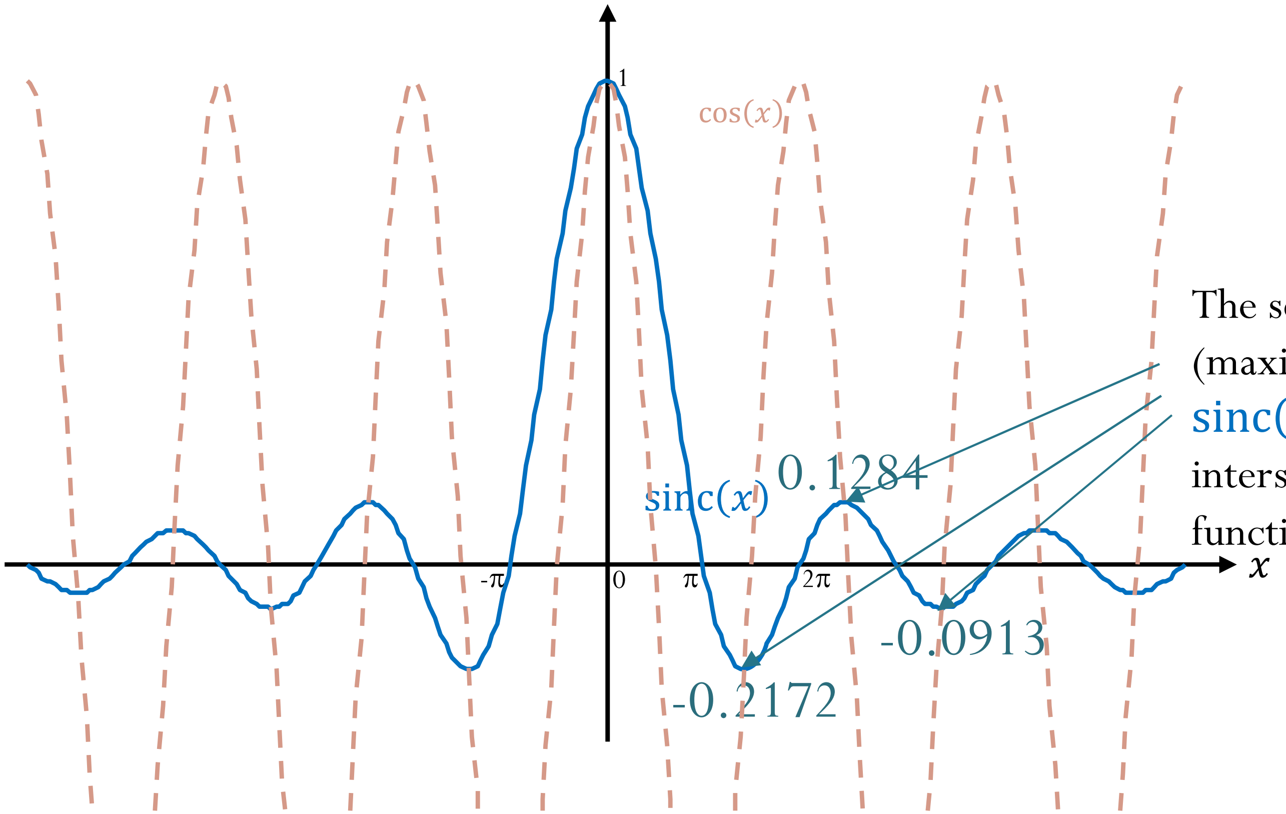
sinc function

Amplitude of $\sin(x)$ decreases continuously as $\frac{1}{x}$.

$$\operatorname{sinc}(x) = \frac{\sin(x)}{x} = \frac{1}{x} \times \sin(x)$$



sinc function



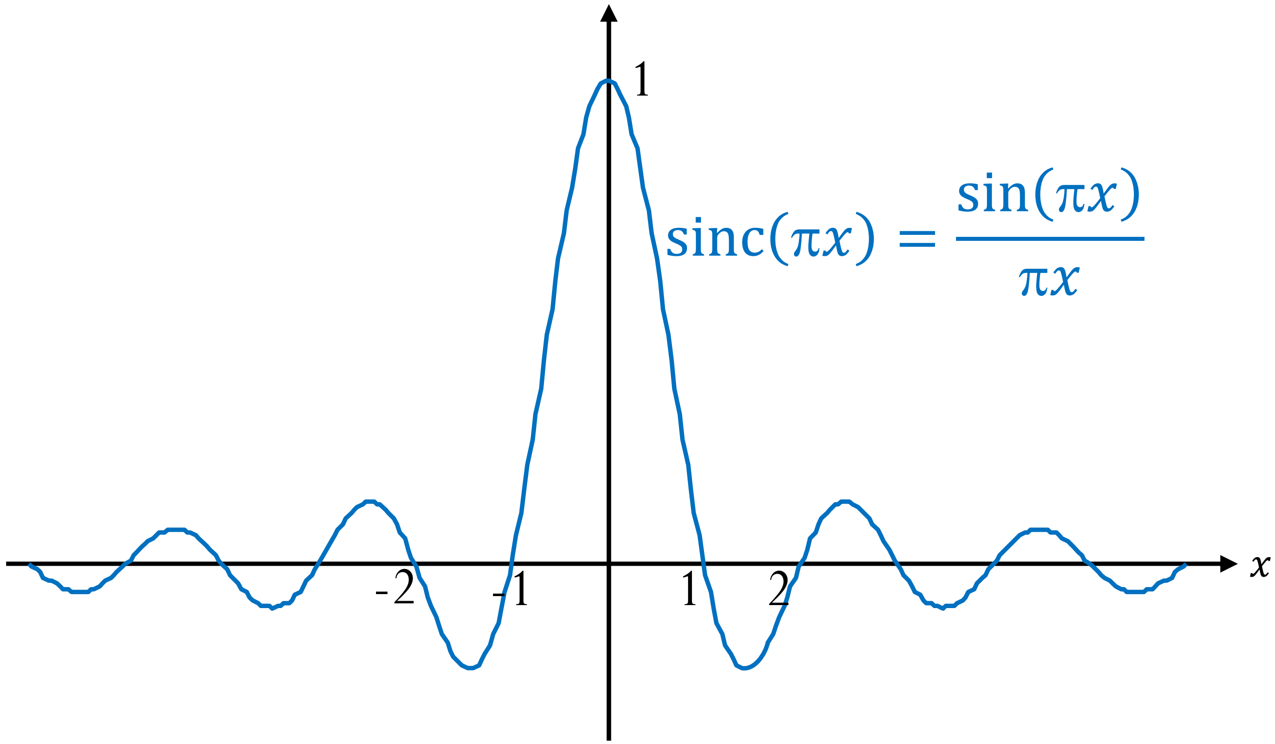
The set of local extrema (maxima and minima) of sinc(x) corresponds to its intersections with the cosine function cos(x)

A good approximation of the x-coordinate of the n-th extremum with positive x-coordinate is

$$x_n \approx (n+\frac{1}{2})\pi - \frac{1}{\left(n+\frac{1}{2}\right)\pi},$$

where odd n lead to a local minimum and even n to a local maximum.

Normalized sinc function



Its zero crossings are at non-zero integer values of its argument.

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Paper No. 1239
RADIO SECTION

INFORMATION THEORY AND INVERSE PROBABILITY IN TELECOMMUNICATION

By P. M. WOODWARD, B.A., and I. L. DAVIES, M.A., Graduate.

(*The paper was first received* 11*th October, and in revised form* 10*th December*, 1951.)

Sam[pling] theorem ... greater than W, then

$$f(t) \equiv \sum_r f(r/2W) \operatorname{sinc}(2Wt - r) \quad . \quad . \quad . \quad (24)$$

where sinc x is an abbreviation for the function $(\sin \pi x)/\pi x$. This function occurs so often in Fourier analysis and its applications that it does seem to merit some notation of its own. Its most important properties are that it is zero when x is a whole number but unity when x is zero, and that

$$\int_{-\infty}^{\infty} \operatorname{sinc} x \, dx = 1$$

and
$$\int_{-\infty}^{\infty} \operatorname{sinc}(x-r)\operatorname{sinc}(x-s)\,dx = \begin{cases} 1, & r = s \\ 0, & r \neq s \end{cases}$$

r and s both being integers. The importance of the identity (24)

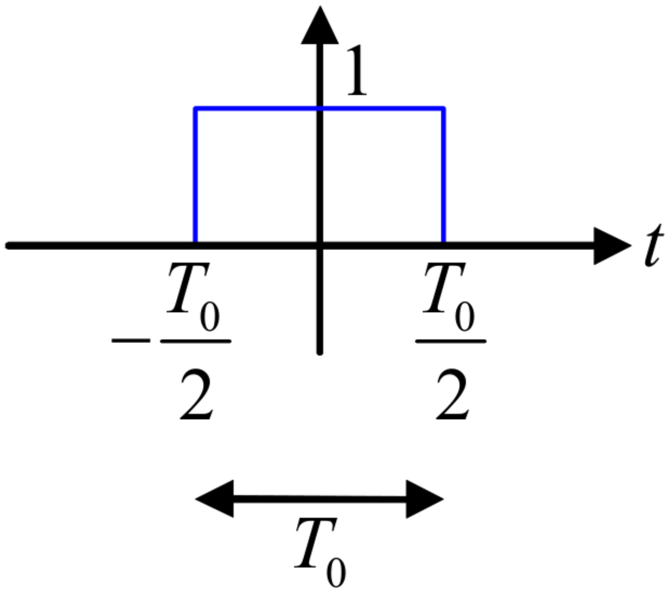
Normalized sinc function

Caution: These are properties of the normalized sinc function. For us, "sinc" is unnormalized. So, the integrations will give different answers.

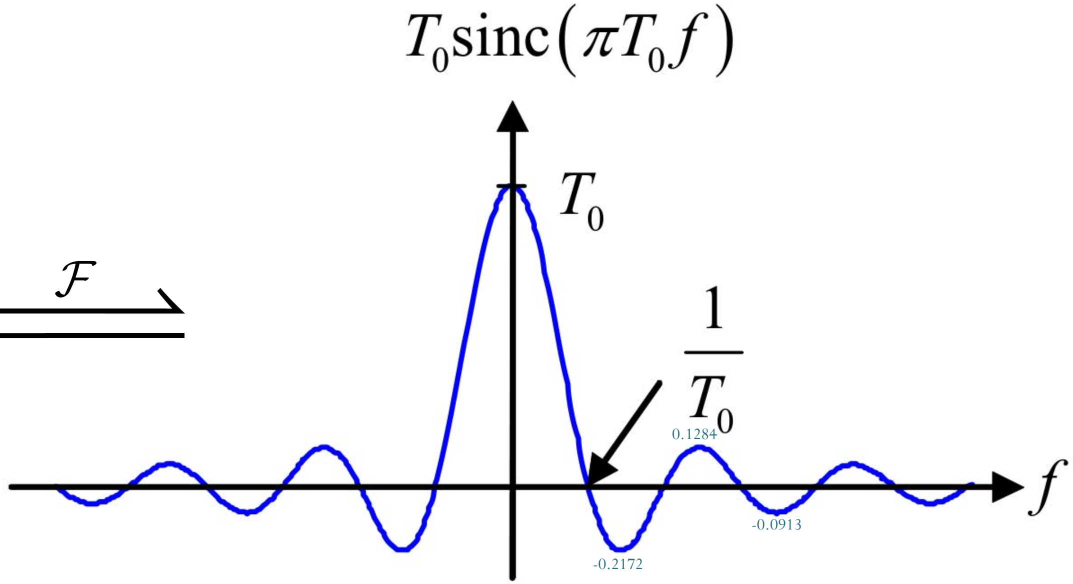
Fourier Transform Pairs (2)

Time Domain | Frequency Domain

$$g(t)=\int_{-\infty}^{\infty} G(f)e^{j2\pi ft}df \overset{\mathcal{F}}{\rightleftharpoons} G(f)=\int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}dt$$



$\overset{\mathcal{F}}{\rightleftharpoons}$



$\text{sinc}(x)=\frac{\sin x}{x}$

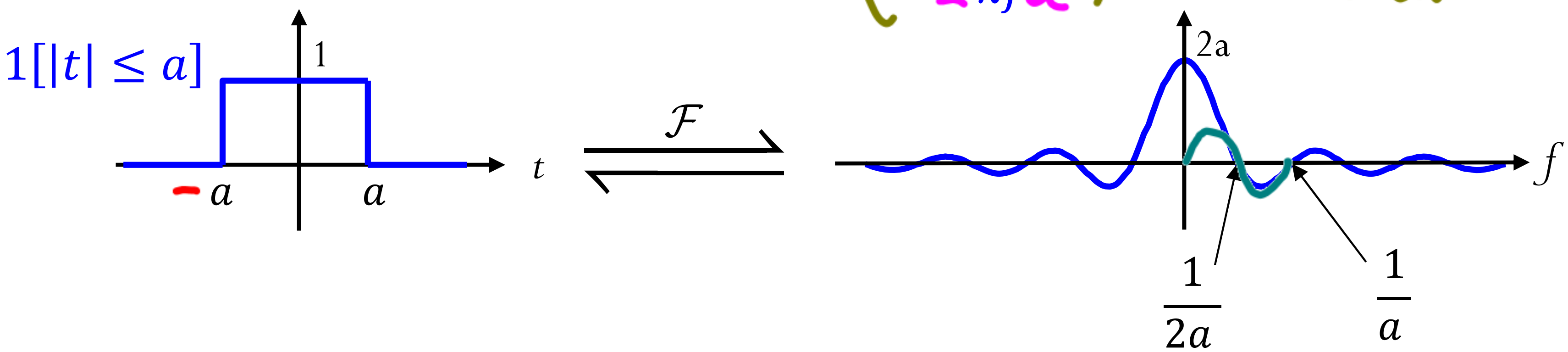
Fourier Transform of Symbolic Rectangular Function in MATLAB

```
>> syms a t
>> g = rectangularPulse(-a,a,t)
g =
rectangularPulse(-a, a, t)
>> G = fourierf(g)
G =
sin(2*pi*a*f)/(pi*f)
```

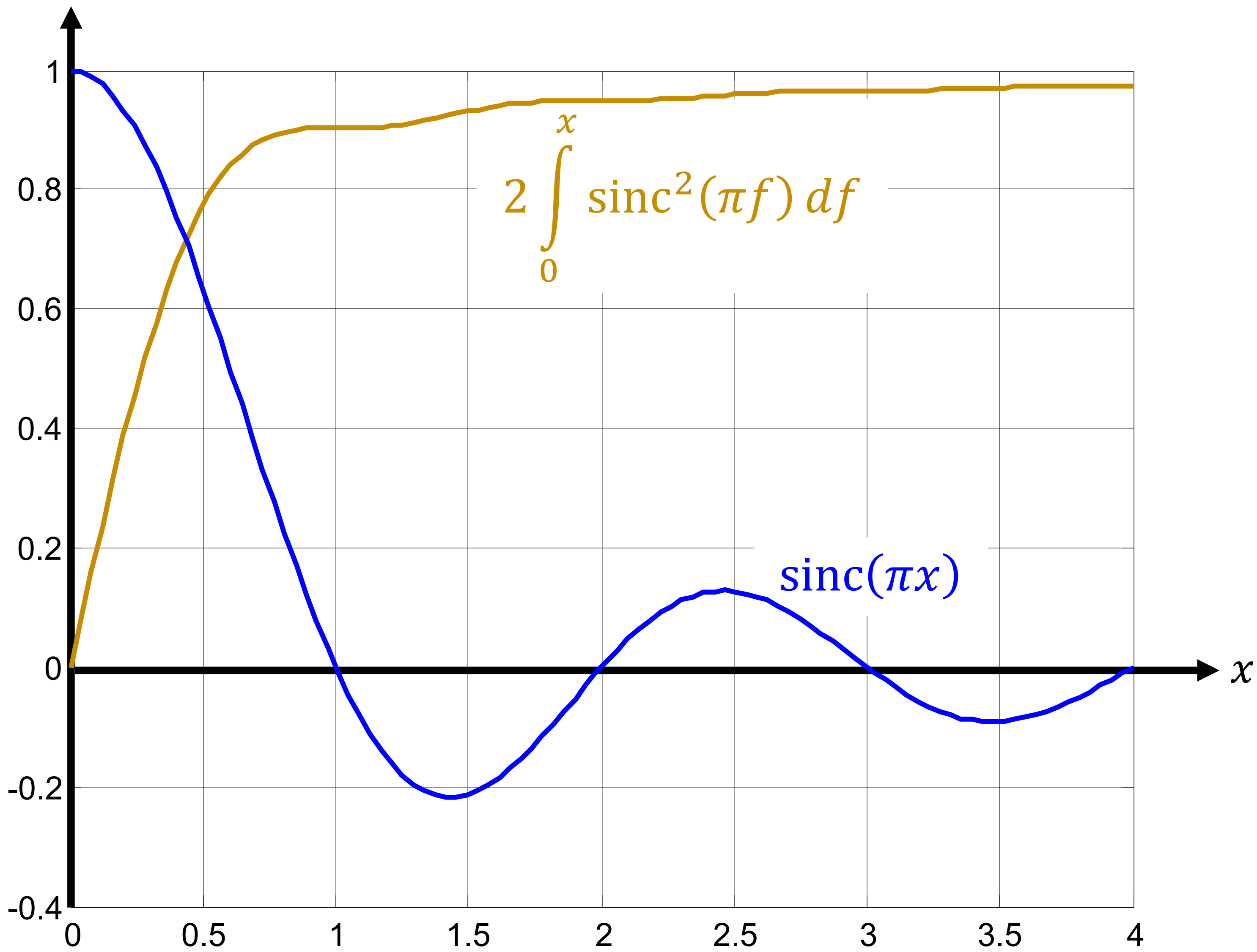
"freq" = a

"period" = $\frac{1}{a}$

$$2a\left(\frac{\sin(2\pi f a)}{2\pi f a}\right) = 2a\,\mathrm{sinc}(2\pi f a)$$



Cumulative Energy

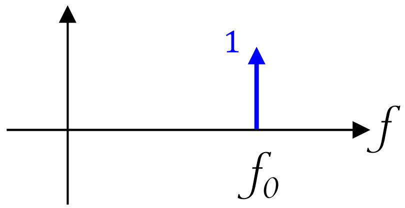


Fourier Transform Pairs (1)

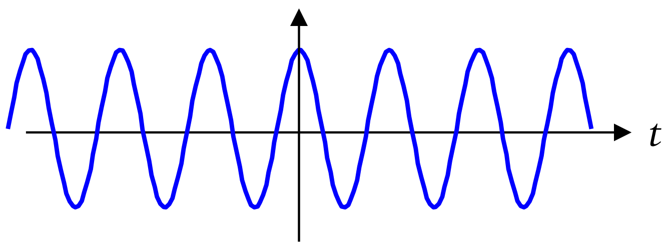
Time Domain | Frequency Domain

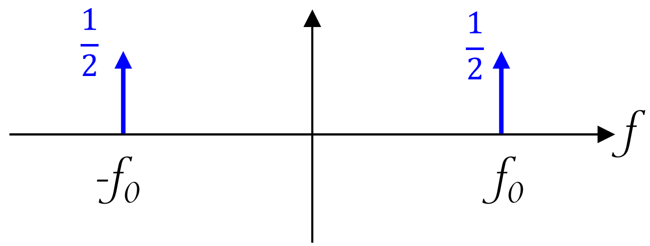
$$g(t)=\int_{-\infty}^{\infty}G(f)e^{j2\pi ft}df \;\overset{\mathcal{F}}{\rightleftharpoons}\; G(f)=\int_{-\infty}^{\infty}g(t)e^{-j2\pi ft}dt$$

$$e^{j2\pi f_0 t} \;\overset{\mathcal{F}}{\rightleftharpoons}\; \delta(f-f_0)$$



$$\frac{e^{j2\pi f_0 t}+e^{-j2\pi f_0 t}}{2}=\cos(2\pi f_0 t) \;\overset{\mathcal{F}}{\rightleftharpoons}\; \frac{1}{2}\delta(f-(-f_0))+\frac{1}{2}\delta(f-f_0)$$

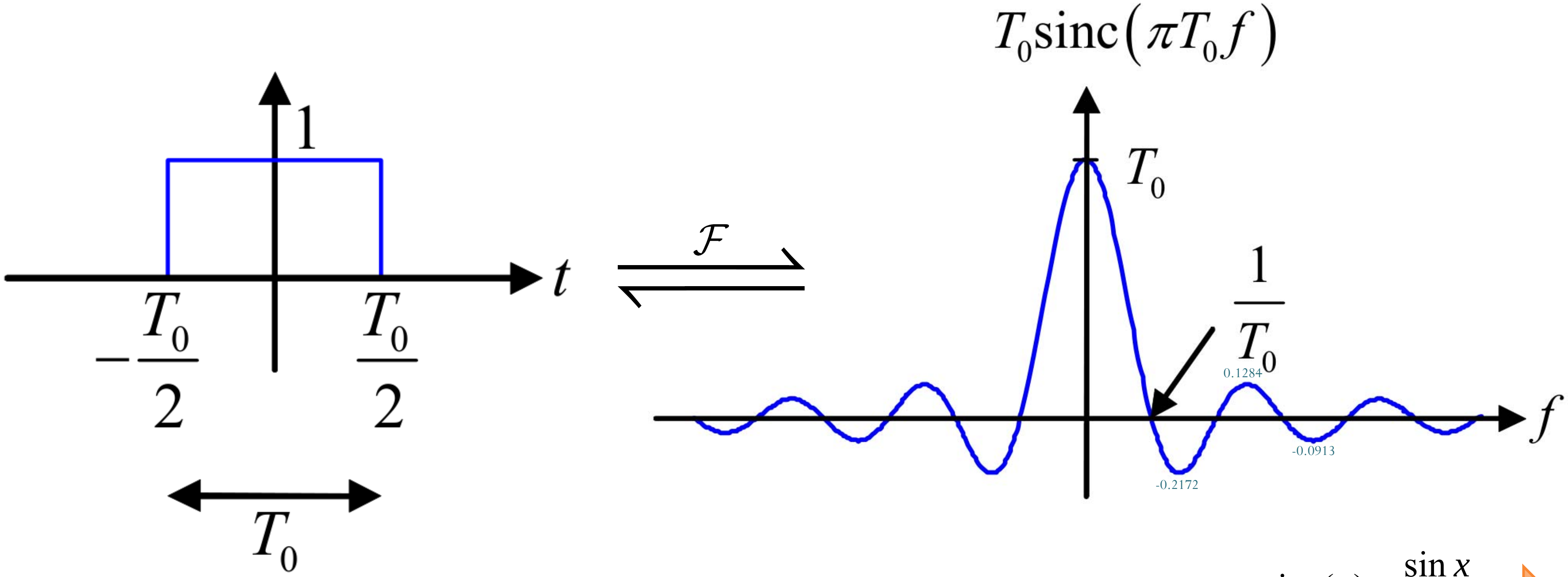




Fourier Transform Pairs (2)

Time Domain — Frequency Domain

$$g(t) = \int_{-\infty}^{\infty} G(f)e^{j2\pi ft}df \overset{\mathcal{F}}{\rightleftharpoons} G(f) = \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}dt$$

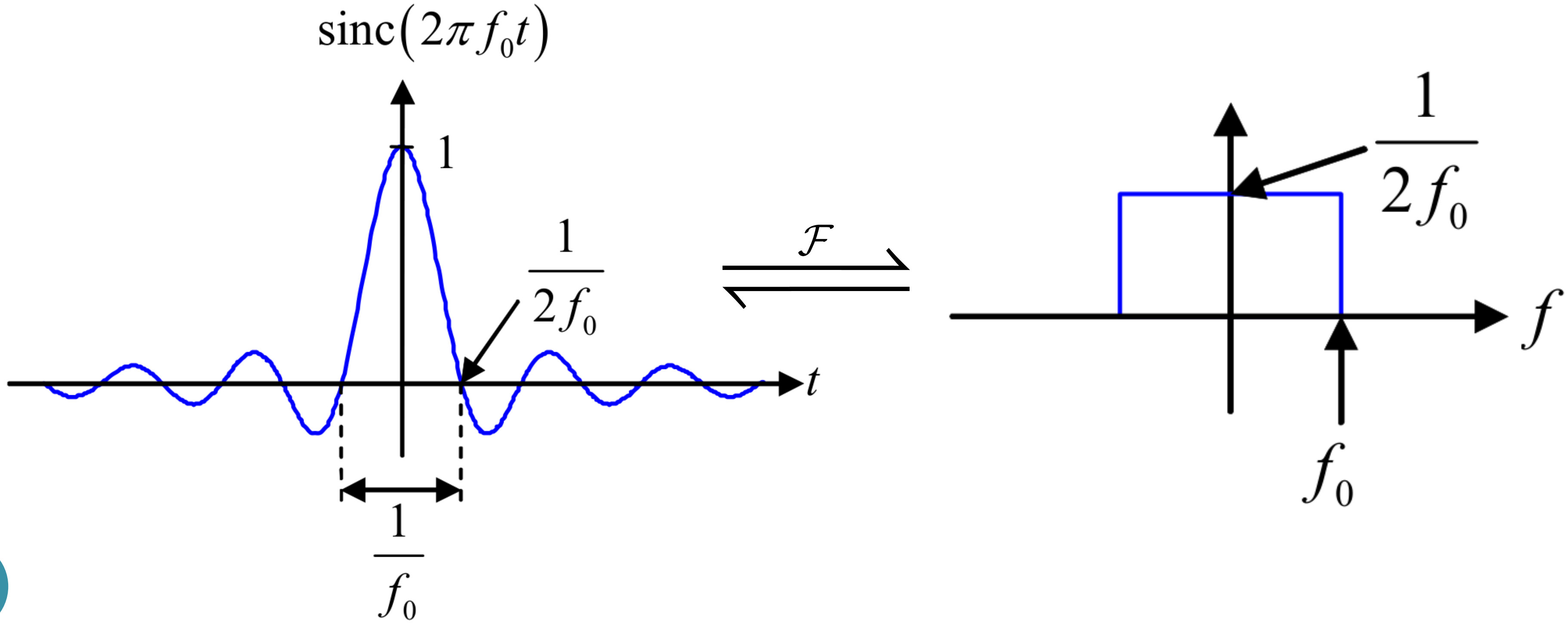


$$\text{sinc}(x) = \frac{\sin x}{x}$$

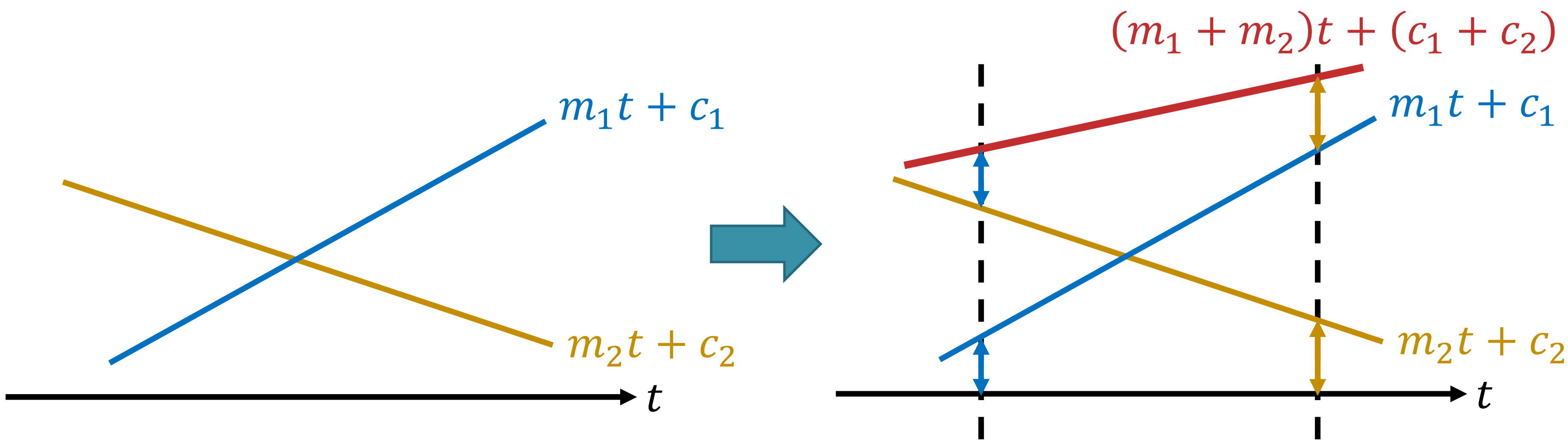
Fourier Transform Pairs (3)

Time Domain　　　　　　　　Frequency Domain

$$g(t)=\int_{-\infty}^{\infty} G(f)e^{j2\pi ft}df \overset{\mathcal{F}}{\rightleftharpoons} G(f)=\int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}dt$$



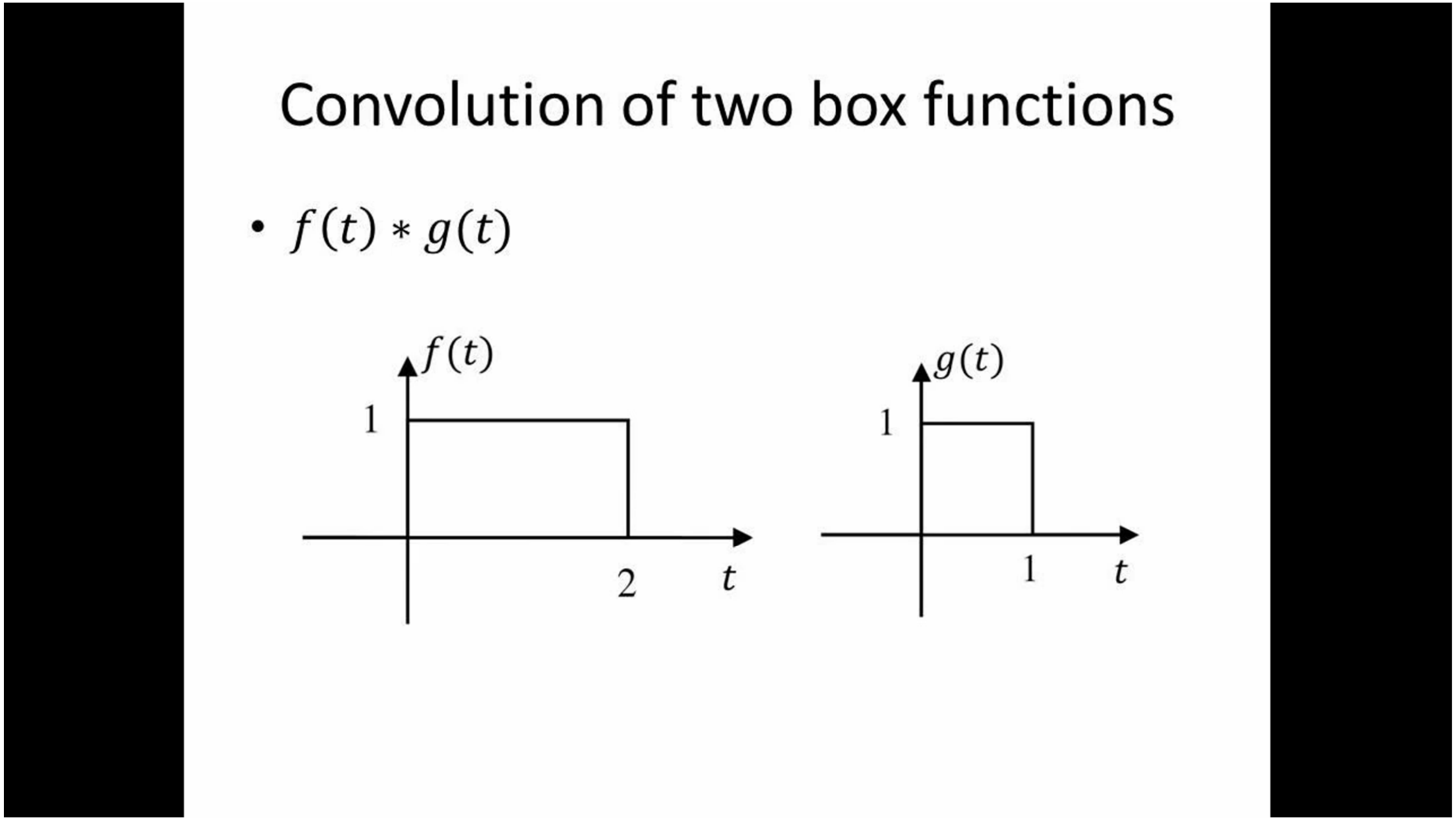
Drawing the sum of two straight lines



$$(m_1 t + c_1) + (m_2 t + c_2) = (m_1 + m_2)t + (c_1 + c_2)$$

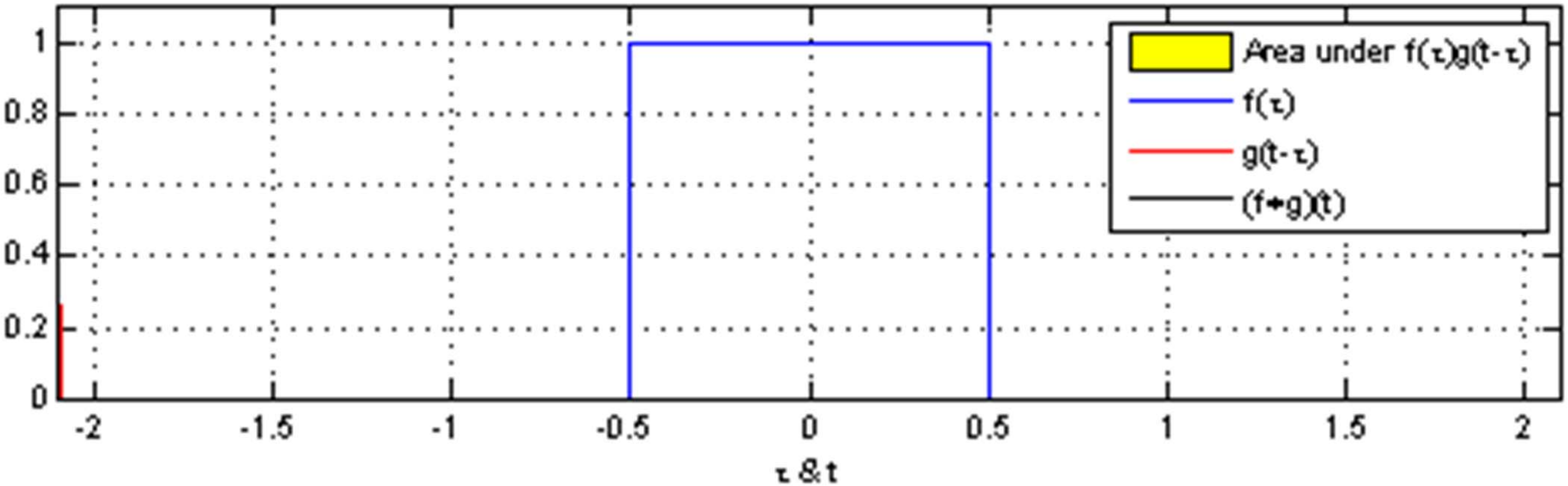
Still a straight line.
So, it's enough to locate two points and then draw a straight line through them.

Convolution: Ex 1



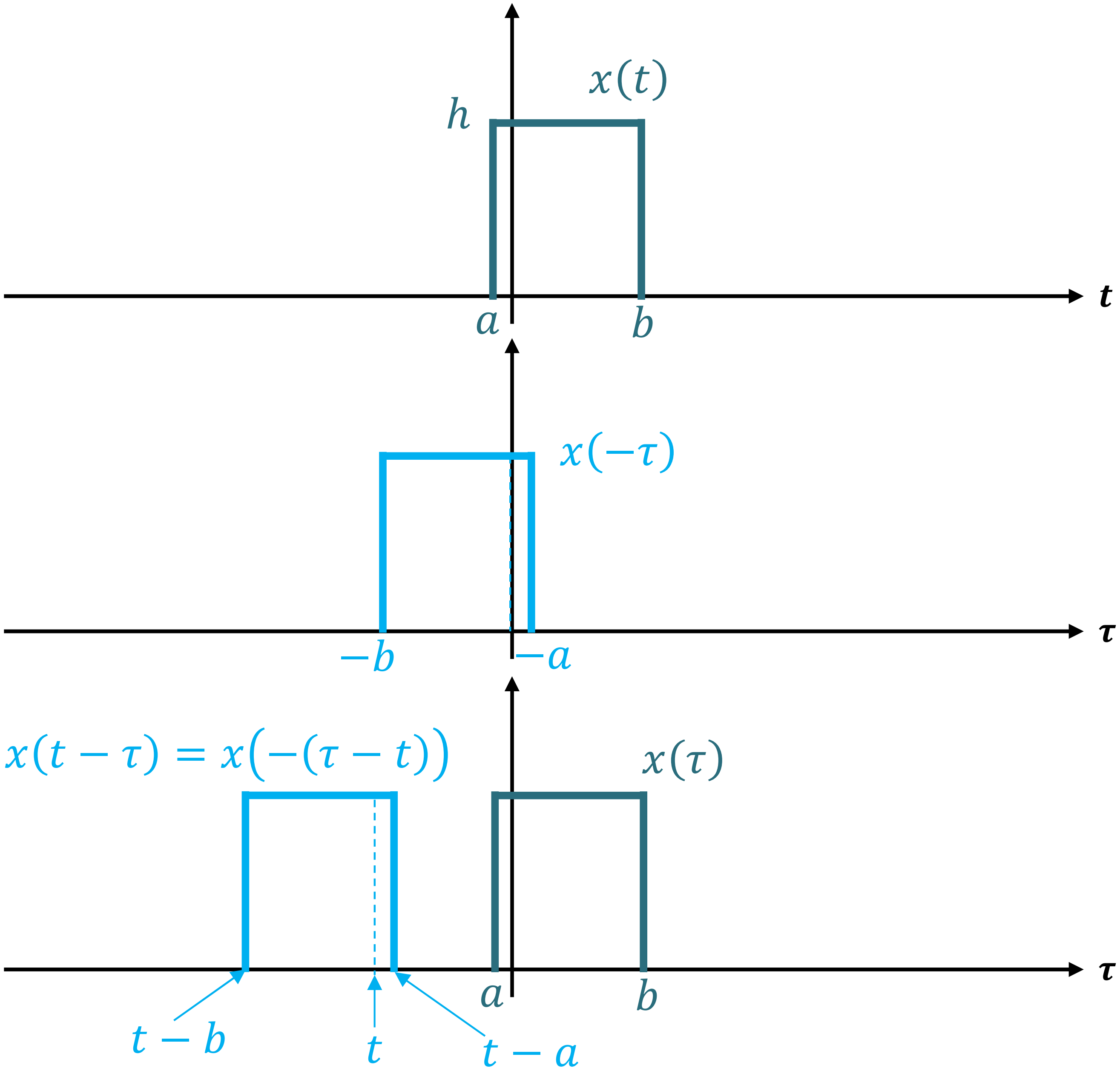
[http://www.youtube.com/watch?v=C1N55M1VD2o]

Convolution: Ex 2

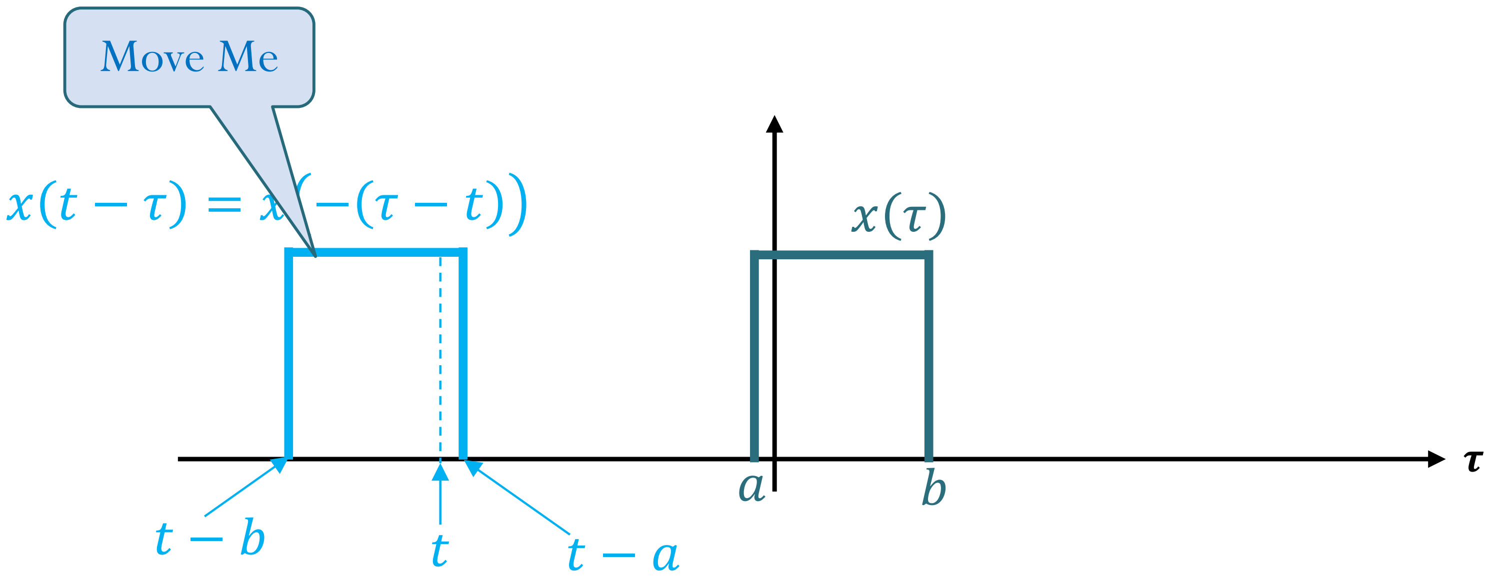


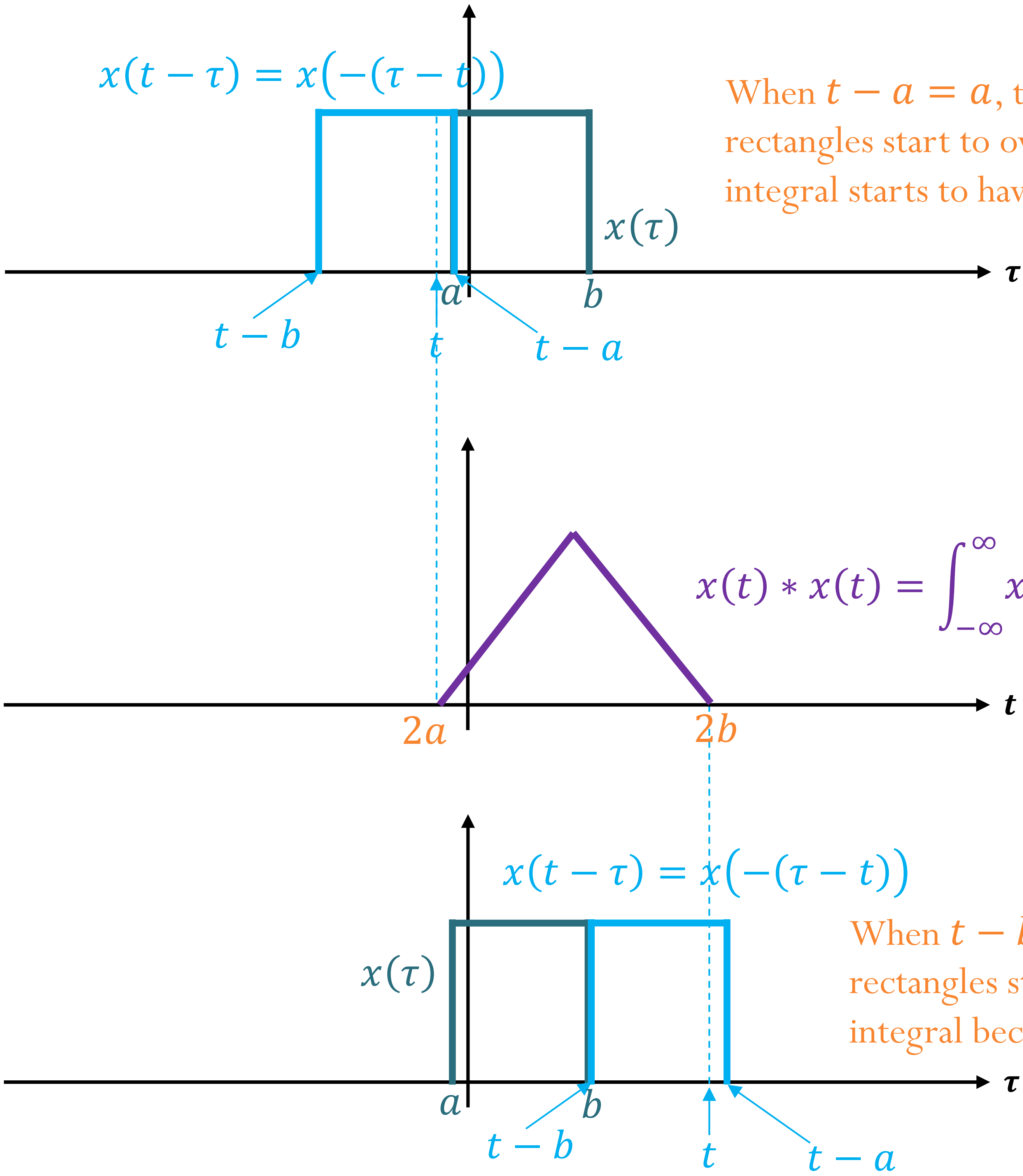
$(x * x)(t)$

Suppose we want to find $x(t) * x(t) = \int_{-\infty}^{\infty} x(\tau)x(t-\tau)d\tau$



Suppose we want to find $x(t) * x(t) = \int_{-\infty}^{\infty} x(\tau)x(t-\tau)d\tau$

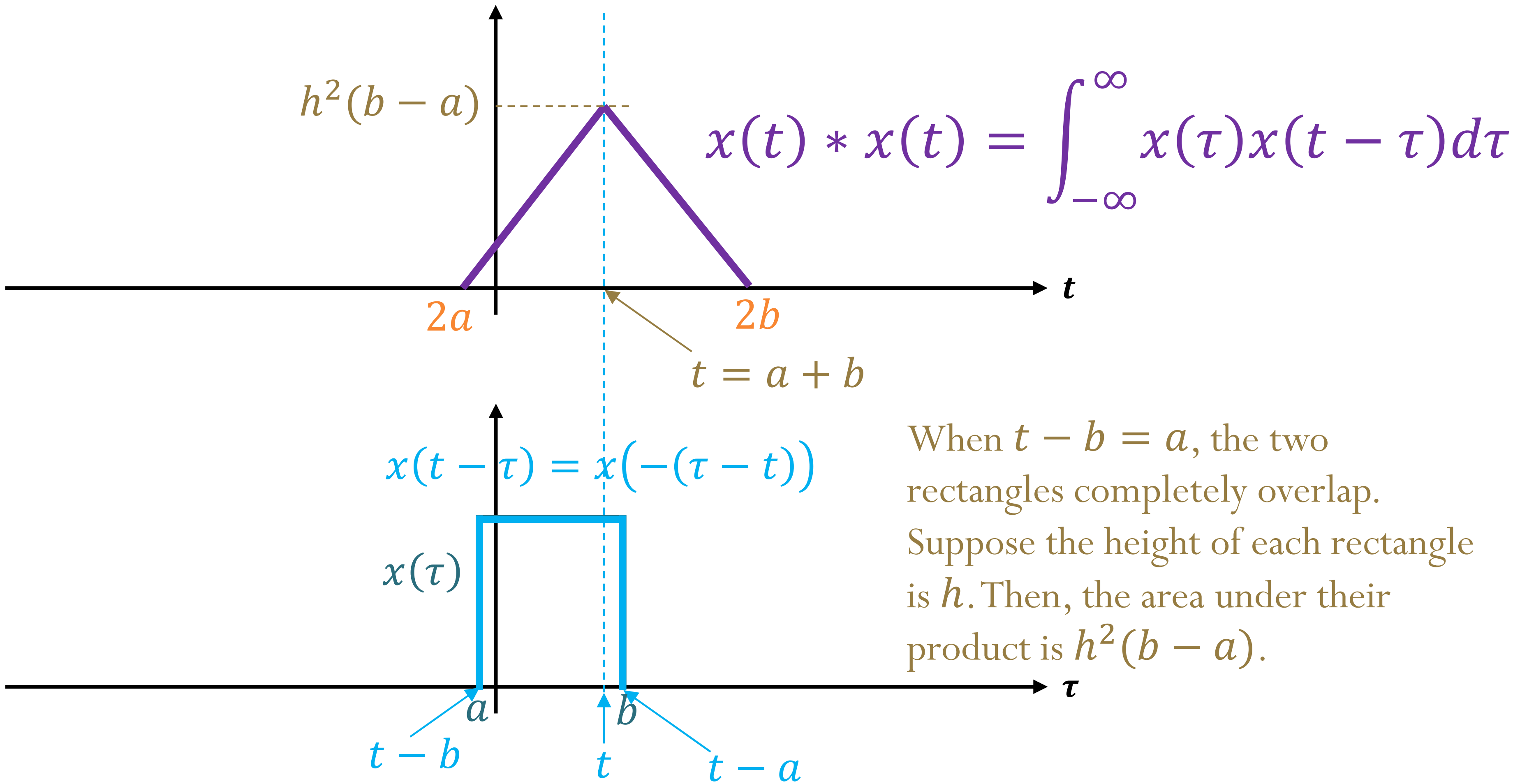




When $t - a = a$, the two rectangles start to overlap and the integral starts to have some value.

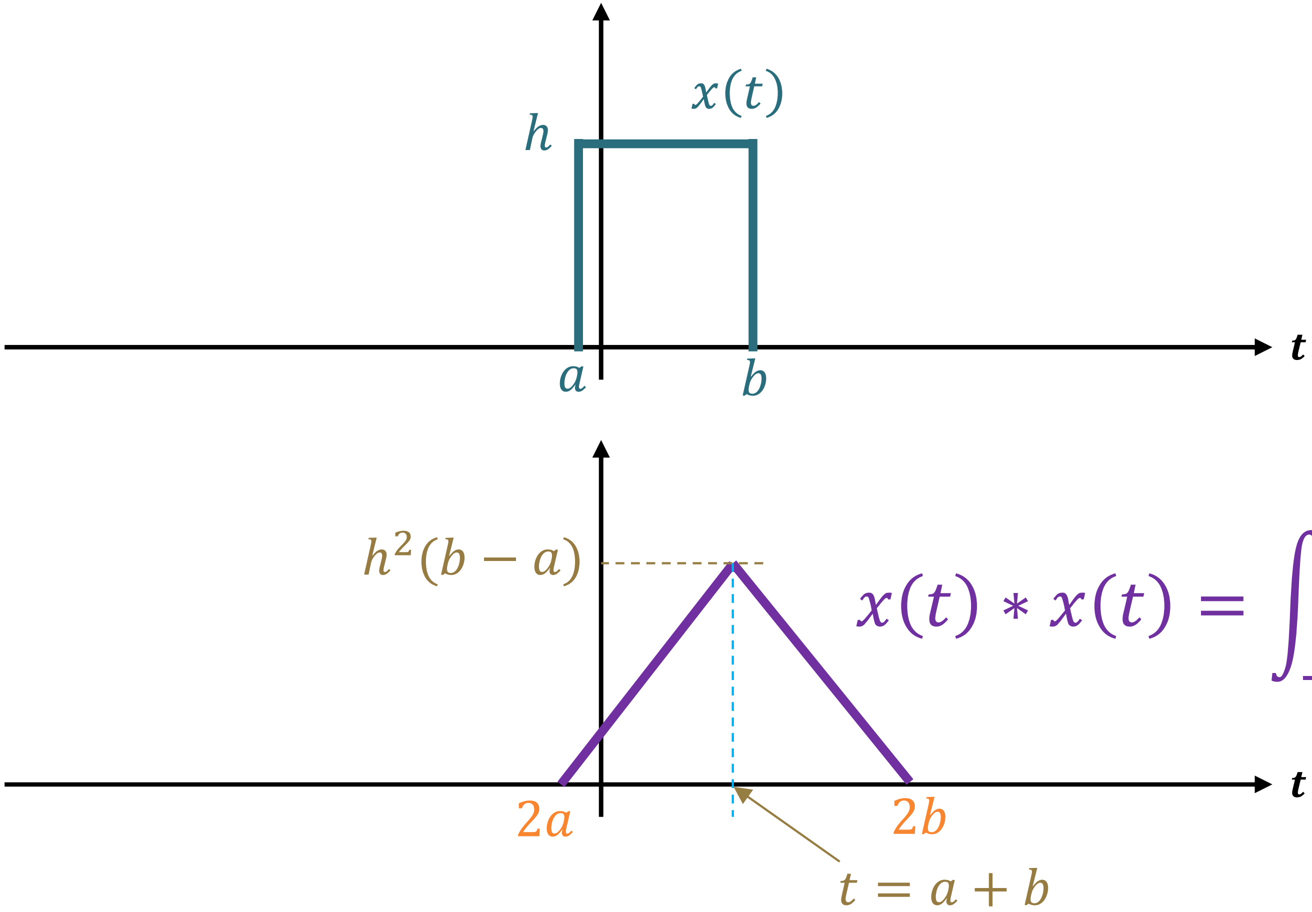
$$x(t) * x(t) = \int_{-\infty}^{\infty} x(\tau)x(t-\tau)d\tau$$

When $t - b = b$, the two rectangles start to separate and the integral becomes 0.



When $t - b = a$, the two rectangles completely overlap. Suppose the height of each rectangle is h. Then, the area under their product is $h^2(b-a)$.

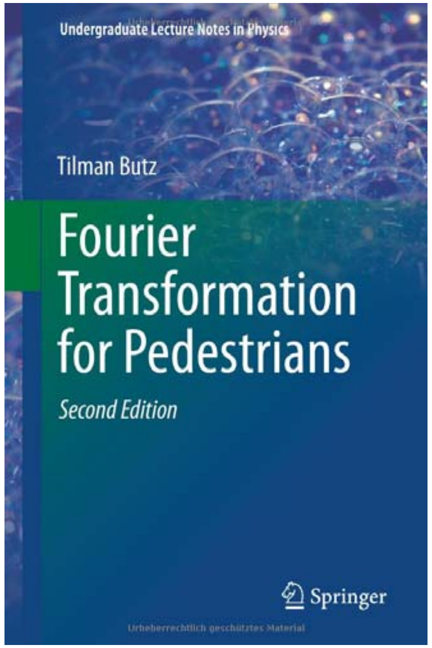
Self-Convolution of Rect. Func.



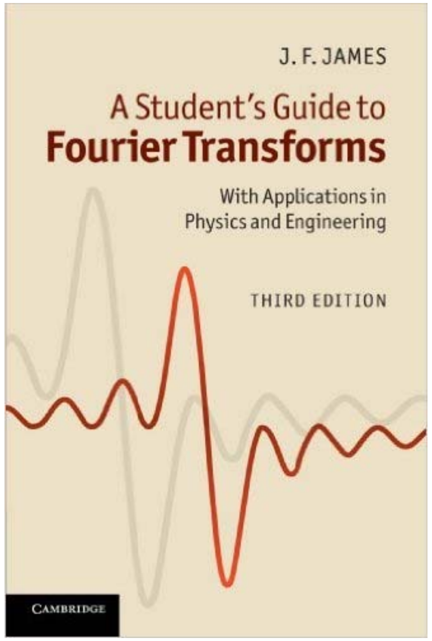
$$x(t) * x(t) = \int_{-\infty}^{\infty} x(\tau)x(t-\tau)d\tau$$

Supplementary References

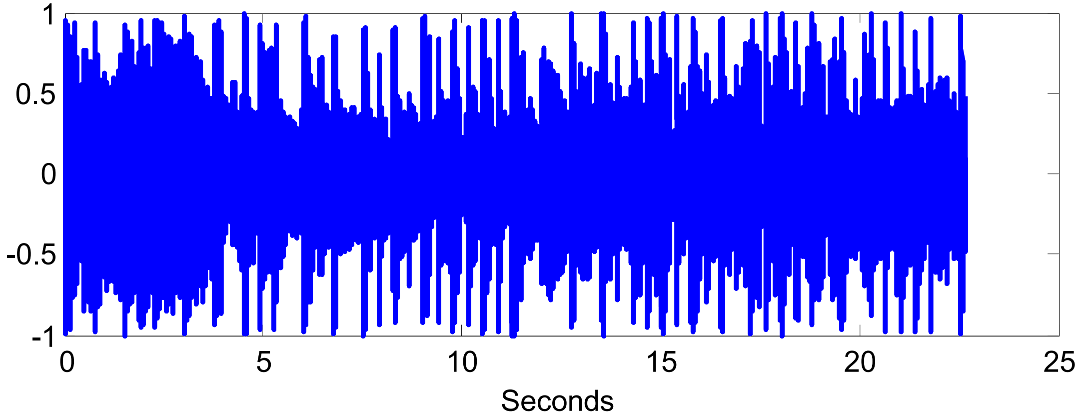
- Fourier Transformation for Pedestrians
  - by Tilman Butz
  - 2^{nd} ed., 2015

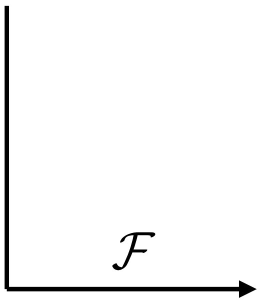


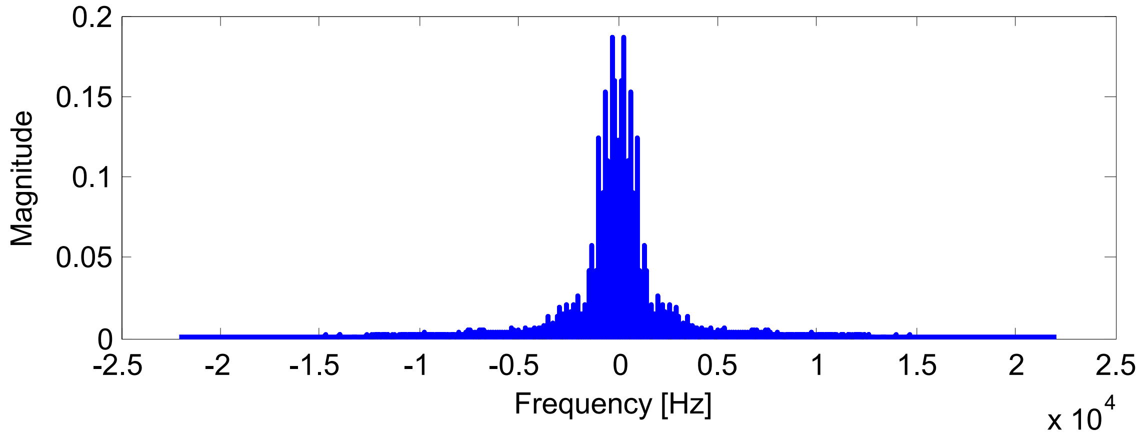
- A Student's Guide to Fourier Transforms
  - by J. F. James
  - 3^{rd} ed., 2011



More realistic signal...







`plotspect.m`

MATLAB scripts

Chapter 2: Frequency-Domain Analysis

Chapter 2: Frequency-Domain Analysis — Edited Sep 9

MATLAB scripts for Chapter 2 — *Scheduled for Tomorrow, 10:...*

- **specrect.m** — Objective C
- **plotspect.m** — Objective C

Edit material

plotspect.m

```
% plotspec(x,t) plots the spectrum of the signal x
% whose values are sampled at time (in seconds) specified in t
function plotspect(x,t)
N=length(x);                                   % length of the signal x
Ts = t(2)-t(1);                                % find the sampling interval
ssf=((-N/2):(N/2-1))/(Ts*N);                   % frequency vector
fx=Ts*fft(x(1:N));                             % do DFT/FFT
fxs=fftshift(fx);                              % shift it for plotting
subplot(2,1,1);
set(plot(t,x),'LineWidth',1.5);                % plot the waveform
xlabel('Seconds');                             % label the axes
subplot(2,1,2);
set(plot(ssf,abs(fxs)),'LineWidth',1.5);       % plot magnitude spectrum
xlabel('Frequency [Hz]'); ylabel('Magnitude')    % label the axes
```

An Example

```
% specrect.m plot the magnitude of the Fourier transform
% of a square wave
close all
Ts=1/100;           % time interval between adjacent samples
t=0:Ts:2e2;         % create a time vector
g=[t <= 2];         % rectangular pulse 1[0 ≤ t ≤ 2]
plotspect(g,t)      % call plotspect to draw spectrum
subplot(2,1,1)
xlim([0,20])        % look only from t
subplot(2,1,2)
xlim([-2,2])        % look only from f
```

$$g(t) = \begin{cases} 1, & 0 \le t \le 2, \\ 0, & \text{otherwise.} \end{cases}$$